

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = 5 \times \frac{1}{2} t^{-1/2} = \frac{5}{2} t^{-1/2} = \frac{5}{2} (5x+9)^{-1/2} = \frac{5}{2(5x+9)^{1/2}}$$

Example 3: Differentiate $y = \sin(2x^3 + 9x)$ with respect to x .

$$\text{Let } t = 2x^3 + 9x$$

$$y = \sin t$$

$$\frac{dt}{dx} = 6x^2 + 9$$

$$\frac{dy}{dt} = \cos t$$

Using the Chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = \cos(t) \times (6x^2 + 9) = (6x^2 + 9)\cos(2x^3 + 9x)$$

Questions (Chain rule):

Differentiate the following functions with respect to x .

1. $y = (3x^2 + 2x)^5$

2. $y = \cos(5x^3 + 6)$

3. $y = e^{x^3+2x^2}$

4. $y = \ln(x^4 + 2x)$

(Solutions on page 8)

The Product Rule

Consider $y = uv$, where u and v are functions of x , then

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

Example 1: Differentiate $y = 7xe^{2x}$

Let $u = 7x$ $v = e^{2x}$

$$\frac{du}{dx} = 7 \quad \frac{dv}{dx} = 2e^{2x}$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\frac{dy}{dx} = 7e^{2x} + 14xe^{2x} = 7e^{2x}(1 + 2x)$$

Example 2: Differentiate $y = x^3 \sin(3x)$

Let $u = x^3$ $v = \sin(3x)$

$$\frac{du}{dx} = 3x^2 \quad \frac{dv}{dx} = 3\cos(3x)$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \sin(3x) + 3x^3 \cos(3x) \\ &= 3x^2(\sin(3x) + x\cos(3x)) \end{aligned}$$

Example 3: Differentiate $y = e^{2x} \ln(5x)$

Let $u = e^{2x}$ $v = \ln(5x)$

$$\frac{du}{dx} = 2e^{2x} \quad \frac{dv}{dx} = \frac{1}{x}$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 2e^{2x} \ln(5x) + \frac{e^{2x}}{x} \\ &= e^{2x} \left(2 \ln(5x) + \frac{1}{x} \right) \end{aligned}$$

Questions (Produce rule):

Differentiate the following functions with respect to x:

1. $y = 5x \ln(6x)$
2. $y = e^{5x}(4x + 2)$
3. $y = 3x^5 \cos(4x)$
4. $y = e^{4x} \ln(7x)$

(Solutions on page 9)

The Quotient Rule

Consider $y = \frac{u}{v}$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Example 1: Differentiate $y = \frac{x^2 - 1}{x^2 + 1}$

$$u = x^2 - 1 \qquad v = x^2 + 1$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2x$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{2x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 + 1)^2}$$

$$\begin{aligned}
 &= \frac{2x[(x^2 + 1) - (x^2 - 1)]}{(x^2 + 1)^2} \\
 &= \frac{2x(x^2 + 1 - x^2 + 1)}{(x^2 + 1)^2} \\
 &= \frac{2x(2)}{(x^2 + 1)^2} \\
 &= \frac{4x}{(x^2 + 1)^2}
 \end{aligned}$$

Example 2: Differentiate $y = \frac{\ln x}{x^2 + 5x}$

$$u = \ln(x) \quad v = x^2 + 5x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 2x + 5$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\frac{1}{x}(x^2 + 5x) - (2x + 5)\ln x}{(x^2 + 5x)^2} \\
 &= \frac{(x + 5) - (2x + 5)\ln x}{(x^2 + 5x)^2}
 \end{aligned}$$

Example 3: Differentiate $y = \frac{5e^x - 1}{3e^x + 2x}$

$$u = 5e^x - 1 \quad v = 3e^x + 2x$$

$$\frac{du}{dx} = 5e^x$$

$$\frac{dv}{dx} = 3e^x + 2$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{5e^x(3e^x + 2x) - (5e^x - 1)(3e^x + 2)}{(3e^x + 2x)^2} \\ &= \frac{15e^{2x} + 10xe^x - (15e^{2x} + 10e^x - 3e^x - 2)}{(3e^x + 2x)^2} \\ &= \frac{15e^{2x} + 10xe^x - 15e^{2x} - 7e^x + 2}{(3e^x + 2x)^2} \\ &= \frac{10xe^x - 7e^x + 2}{(3e^x + 2x)^2} \end{aligned}$$

Questions (Quotient rule):

Differentiate the following functions with respect to x:

1. $y = \frac{x^2}{1-x}$

2. $y = \frac{4x^3}{\cos x}$

3. $y = \frac{\ln(x)}{x^2 - 1}$

4. $y = \frac{e^{2x}}{\sin x + \cos x}$

(Solutions on page 10)

Combinations

Sometimes you may need to use a combination of the rules to differentiate a function.

Example 1: Differentiate $y = x^5(x + 3)^4$

Solution:

Let $u = x^5$

$$\frac{du}{dx} = 5x^4$$

$v = (x + 3)^4$

$$\frac{dv}{dx} = 4(x + 3)^3$$

Using the chain rule:
 $t = x + 3 \quad v = t^4$
 $\frac{dt}{dx} = 1 \quad \frac{dv}{dt} = 4t^3$

Using the chain rule:
 $\frac{dv}{dx} = \frac{dt}{dx} \times \frac{dv}{dt}$
 $= 1 \times 4t^3$
 $= 4(x + 3)^3$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 5x^4(x+3)^4 + 4x^5(x+3)^3 \\ &= x^4(x+3)^3[5(x+3) + 4x] \\ &= x^4(x+3)^3[5x+15+4x] \\ &= x^4(x+3)^3(9x+15) \end{aligned}$$

Example 2: Differentiate $y = \frac{\ln(x^2+1)}{x^2-1}$

Solution:

$$u = \ln(x^2+1) \quad v = x^2-1$$

$$\frac{du}{dx} = \frac{2x}{x^2+1} \quad \frac{dv}{dx} = 2x$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x^2-1)\left(\frac{2x}{x^2+1}\right) - \ln(x^2+1) \times (2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2-1) - 2x(x^2+1)\ln(x^2+1)}{(x^2+1)(x^2-1)^2}$$

Using the chain rule:
 $t = x^2 + 1 \quad u = \ln t$
 $\frac{dt}{dx} = 2x \quad \frac{du}{dt} = \frac{1}{t}$

Using the chain rule:
 $\frac{du}{dx} = \frac{dt}{dx} \times \frac{du}{dt}$
 $= 2x \times \frac{1}{t}$
 $= \frac{2x}{t} = \frac{2x}{x^2+1}$

Questions (Combinations):

Differentiate the following functions with respect to x:

1. $y = (2x^2 + 3)(3x + 4)^3$

2. $y = \frac{x^4}{(x+1)^2}$

(Solutions on page 12)

Solutions (Chain rule):

1. $y = (3x^2 + 2x)^5$
 Let $t = 3x^2 + 2x$ $y = t^5$
 $\frac{dt}{dx} = 6x + 2$ $\frac{dy}{dt} = 5t^4$
 Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
 $\frac{dy}{dx} = 5t^4 \times (6x + 2)$
 $= 5t^4(6x + 2) = 5(3x^2 + 2x)^4(6x + 2)$

2. $y = \cos(5x^3 + 6)$
 Let $t = 5x^3 + 6$ $y = \cos t$
 $\frac{dt}{dx} = 15x^2$ $\frac{dy}{dt} = -\sin t$
 Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
 $\frac{dy}{dx} = -\sin t \times (15x^2) = -15x^2 \sin(t) = -15x^2 \sin(5x^3 + 6)$

3. $y = e^{x^3 + 2x^2}$
 Let $t = x^3 + 2x^2$ $y = e^t$
 $\frac{dt}{dx} = 3x^2 + 4x$ $\frac{dy}{dt} = e^t$
 Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
 $\frac{dy}{dx} = (3x^2 + 4x) e^{x^3 + 2x^2}$

4. $y = \ln(x^4 + 2x)$
 Let $t = x^4 + 2x$ $y = \ln t$
 $\frac{dt}{dx} = 4x^3 + 2$ $\frac{dy}{dt} = \frac{1}{t}$
 Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= (4x^3 + 2) \times \frac{1}{t} \\ &= \frac{(4x^3 + 2)}{(x^4 + 2x)}\end{aligned}$$

Solutions (Product rule):

1. $y = 5x \ln(6x)$

Let $u = 5x$ $v = \ln(6x)$

$$\frac{du}{dx} = 5 \quad \frac{dv}{dx} = \frac{1}{x}$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 5 \ln(x) + 5x \left(\frac{1}{x} \right) \\ &= 5 \ln(x) + 5 \\ &= 5(\ln(x) + 1)\end{aligned}$$

2. $y = e^{5x}(4x + 2)$

Let $u = e^{5x}$ $v = 4x + 3$

$$\frac{du}{dx} = 5e^{5x} \quad \frac{dv}{dx} = 4$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 5e^{5x}(4x + 3) + 4e^{5x} \\ &= e^{5x}(5(4x + 3) + 4) \\ &= e^{5x}(20x + 15 + 4) \\ &= e^{5x}(20x + 19)\end{aligned}$$

3. $y = 3x^5 \cos(4x)$

Let $u = 3x^5$ $v = \cos(4x)$

$$\frac{du}{dx} = 15x^4 \quad \frac{dv}{dx} = -4 \sin(4x)$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 15x^4 \cos(4x) - 12x^5 \sin(4x) \\ &= 3x^4 [5\cos(4x) - 4x\sin(4x)] \end{aligned}$$

4. $y = e^{4x} \ln(7x)$

Let $u = e^{4x}$ $v = \ln(7x)$

$$\frac{du}{dx} = 4e^{4x} \qquad \frac{dv}{dx} = \frac{1}{x}$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 4e^{4x} \ln(7x) + \frac{e^{4x}}{x} \\ &= e^{4x} \left(4\ln(7x) + \frac{1}{x} \right) \end{aligned}$$

Solutions (Quotient rule):

1. $y = \frac{x^2}{1-x}$

$u = x^2$ $v = 1 - x$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = -1$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x(1-x) - x^2(-1)}{(1-x)^2} \\ &= \frac{2x - 2x^2 + x^2}{(1-x)^2} \\ &= \frac{2x - x^2}{(1-x)^2} \\ &= \frac{x(2-x)}{(1-x)^2} \end{aligned}$$

$$2. y = \frac{4x^3}{\cos x}$$

$$u = 4x^3 \quad v = \cos(x)$$

$$\frac{du}{dx} = 12x^2 \quad \frac{dv}{dx} = -\sin x$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{12x^2 \cos x + 4x^3 \sin x}{(\cos x)^2}$$

$$= \frac{4x^3(3\cos x + x\sin x)}{\cos^2 x}$$

$$3. y = \frac{\ln(x)}{x^2 - 1}$$

$$u = \ln(x) \quad v = x^2 - 1$$

$$\frac{du}{dx} = \frac{1}{x} \quad \frac{dv}{dx} = 2x$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x^2 - 1)\left(\frac{1}{x}\right) - 2x\ln(x)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{(x - \frac{1}{x}) - 2x\ln(x)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 1 - 2x^2\ln(x)}{x(x^2 - 1)^2}$$

$$4. y = \frac{e^{2x}}{\sin x + \cos x}$$

$$u = e^{2x} \qquad v = \sin(x) + \cos(x)$$

$$\frac{du}{dx} = 2e^{2x} \qquad \frac{dv}{dx} = \cos(x) - \sin(x)$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(\sin x + \cos x)(2e^{2x}) - (e^{2x})(\cos x - \sin x)}{(\sin x + \cos x)^2} \\ &= \frac{e^{2x} [2(\sin x + \cos x) - (\cos x - \sin x)]}{(\sin x + \cos x)^2} \\ &= \frac{e^{2x} (2\sin x + 2\cos x - \cos x + \sin x)}{(\sin x + \cos x)^2} \\ &= \frac{e^{2x} (3\sin x + \cos x)}{(\sin x + \cos x)^2} \end{aligned}$$

Solutions (Combinations):

$$1. y = (2x^2 + 3)(3x + 4)^3$$

$$\text{Let } u = (2x^2 + 3) \qquad v = (3x + 4)^3$$

$$\begin{aligned} \frac{du}{dx} &= 4x & \frac{dv}{dx} &= 3(3x + 4)^2 \cdot 3 \\ & & &= 9(3x + 4)^2 \end{aligned}$$

Using the product rule: $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 4x(3x + 4)^3 + 9(2x^2 + 3)(3x + 4)^2 \\ &= (3x + 4)^2 [4x(3x + 4) + 9(2x^2 + 3)] \\ &= (3x + 4)^2 [12x^2 + 16x + 18x^2 + 27] \\ &= (3x + 4)^2 [30x^2 + 16x + 27] \end{aligned}$$

$$2. \quad y = \frac{x^4}{(x+1)^2}$$

$$\text{Using the quotient rule: } \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$u = x^4 \qquad v = (x+1)^2$$

$$\frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = 2(x+1)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\ &= \frac{4x^3(x+1)^2 - 2x^4(x+1)}{[(x+1)^2]^2} \\ &= \frac{4x^3(x+1)^2 - 2x^4(x+1)}{(x+1)^4} \\ &= \frac{4x^3(x+1) - 2x^4}{(x+1)^3} \\ &= \frac{4x^4 + 4x^3 - 2x^4}{(x+1)^3} \\ &= \frac{2x^4 + 4x^3}{(x+1)^3} \\ &= \frac{2x^3(x+2)}{(x+1)^3} \end{aligned}$$